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Entanglement entropy and spatial geometry

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Abstract: The entanglement entropy in a quantum field theory between two regions of space has been shown in simple cases to be proportional to the volume of the hypersurface separating the regions. We prove that this is true for a free scalar field in an arbitrary geometry with purely spatial curvature and obtain a complete asymptotic expansion for the entropy.

Keywords: AdS-CFT Correspondence, Field Theories in Higher Dimensions.

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Contents

1. Introduction

Modern developments in efforts to consistently combine gravity and quantum mechanics have indicated that quantum field theory has too many degrees of freedom. The entropy of a geometrical object seems to depend on the area of a boundary surface. In the case of a black hole, thermodynamic arguments suggest that its entropy is proportional to the area of its event horizon. This expectation has been confirmed on the microscopic side by calculations for some special cases of black holes.

The separation of a system into two subsystems gives rise to the notion of the entanglement entropy which quantifies the quantum correlations between the two subsystems. One can take the subsystems to be two regions of space separated by a boundary surface. Early calculations in quantum field theories indicated that the entanglement entropy between the degrees of freedom in two regions separated by a boundary surface is proportional to the area of the boundary surface [1 − 4]. These developments have led to the intuition that the entanglement entropy is dominated by degrees of freedom close to the boundary surface, so it is natural to expect that the entanglement entropy is proportional to the area. The connection between the entropy of a geometric system, on the one hand, and the entanglement entropy between the quantum field theory degrees of freedom between spatially disconnected regions, on the other hand, is not clear. There have even been suggestions that the black hole entropy is entirely entanglement entropy [5]. This suggests the possibility that when the boundary surface is taken to be an event horizon, the two types of entropy are identical or at least related. Therefore it becomes important to examine the properties of entanglement entropy in more general cases and attempt to understand its properties when the entanglement involves regions separated by a horizon.

The entanglement entropy is ultraviolet divergent and must be regularized. Presumably a more fundamental theory at the Planck scale will provide the mechanism that eliminates the divergence. Since we are not yet aware of how the underlying theory regularizes the divergence, we are forced to do so by hand in calculations. The regularization involves the Planck scale and the expectation that entanglement entropy may be connected to the theory of quantum gravity is suggested by simple dimensional analysis. If indeed the entanglement entropy is proportional to the area of the boundary surface in Planck units, then the connection to a holographic principle similar to the one suggested by the AdS/CFT correspondence [6 − 10], may apply to the entanglement entropy.

After the initial studies of the entanglement entropy [1 − 4], most research in the subject has been devoted to CFTs. When a field theory is conformal, there are additional tools to compute the entanglement entropy, which, in fact, is proportional to the central charge [11]. Even when the CFT is modified by adding mass deformations [11 − 13], the same property holds. Also, recent calculations of entanglement entropy in this context suggest a holographic interpretation [14, 15]. The connection between the entanglement entropy and holography has also been discussed in more general contexts [16, 17]. Scaling of the entaglement entropy with the area of the boundary have also been verified by numerical computations [18, 19].

In this paper, we investigate the entanglement entropy of a quantum field theory in the case of an arbitrary boundary surface embedded in a background with spatial curvature. Our results contain many previously derived results as special cases, but are more general because we consider arbitrary geometries. Ultimately one would want to consider the even more general case involving spacetime curvature, so the comparison with other forms of entropy can be made explicit.

2. Entanglement entropy

In this section, we compute the entanglement entropy by using two main tools, the replica method and the heat kernel method.

In the replica method (see, for example, ref. [3]), the entropy is expressed as the limit $k \to 1$ of an expression involving the kth power of the (reduced) density matrix. Expressing the density matrix as a path integral, we are lead to consider the manifold which is the result of gluing k copies of the original manifold. This gives an expression of the entropy in terms of the path integral over closed curves in the glued manifold, which is described in the subsection 2.1.

For a free scalar field on an arbitrary base manifold, the resulting path integral is expressed in terms of the spectral quantities of a differential operator. It is convenient to study these quantities by the heat kernel method. We use it in the subsection 2.2 to obtain an expression for the entropy in which the dependence on the hypersurface is factored out.

This leads to several properties of the entropy which we derive in the subsection 2.3. In particular, we compute the entropy for a hypersurface which is a direct product of manifolds, prove the addititivity property of the entropy, and compute the leading terms of asymptotic expansions of the entropy.

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In the heat kernel method (see, for example, refs. [20, 21]), the spectral information is obtained from an asymptotic expansion of the trace of the heat kernel of the operator. All terms in the expansion are determined by the geometry of the underlying manifold. Although they can be computed in principle, the computations are quite complicated in practice. The parameters in the resulting asymptotic expansion are an ultraviolet cutoff scale and geometric scales associated with the manifold. In the subsection 2.4, this leads to an asymptotic expansion for the entropy, which involves geometric quantities associated with the hypersurface. We show that the term proportional to the volume of the manifold is absent and the leading term is proportional to the volume of the hypersurface.

2.1 Replica method

We consider a field theory on a generally curved space which is divided by an arbitrary hypersurface into two parts. The quantum fields in the two parts are entangled, and our goal is the calculation of the entanglement entropy. Let M be an $n-1$ dimensional Riemannian manifold without a boundary and let $\Sigma \subset M$ be a closed hypersurface (a submanifold of codimension 1). Σ divides M into two parts, the interior part M' and the exterior part M''. Let ϕ be a field on M, and (ϕ', ϕ'') its restrictions to (M', M''), and let $\psi(\phi', \phi'')$ be a wave function corresponding to the field having the value (ϕ', ϕ'') on (M', M''). The density matrix for (ϕ', ϕ'') is

$$\rho(\phi'_1, \phi''_1, \phi'_2, \phi''_2) = \psi(\phi'_1, \phi''_1)\overline{\psi(\phi'_2, \phi''_2)}, \tag{2.1}$$

and the reduced density matrix for ϕ' is obtained by tracing over the degrees of freedom of the field on M'',

$$\rho'(\phi'_1, \phi'_2) = \int d\phi''\, \rho(\phi'_1, \phi'', \phi'_2, \phi''). \tag{2.2}$$

The reduced density matrix ρ' represents the mixed state with the associated entropy

$$S' = -\operatorname{tr}\left(\frac{\rho'}{\operatorname{tr}\rho'}\log\frac{\rho'}{\operatorname{tr}\rho'}\right) = \lim_{k\to 1}\left(1 - \frac{\partial}{\partial k}\right)\log\operatorname{tr}\rho'^k. \tag{2.3}$$

The quantity $\operatorname{tr}\rho' = \int d\phi'\, \rho'(\phi', \phi')$ in the denominator guarantees the correct normalization for the density matrix. The second equality embodies the replica method (see, for example, ref. [3]).

In order to obtain the path integral representation for the kth power of the density matrix we introduce an auxiliary field $\varphi(\tau, x)$ defined on $N = \mathbb{R}\times M$ and which satisfies the boundary condition $\varphi(0, x) = \phi_0(x)$. The parameter τ represents Euclidean time. Let $I(\varphi)$ be an action for the field φ. The wave function is $\psi(\phi) = Z(N, \phi_0, \phi)$, where

$$Z(N, \phi_0, \phi) = \int_{C(N,\phi_0,\phi)} d\varphi\, \exp\left(-I(\varphi)\right) \tag{2.4}$$

is a path integral over the space $C(N, \phi_0, \phi)$ of curves defined on N and which satisfy boundary conditions $\varphi(0, x) = \phi_0(x)$ and $\varphi(T, x) = \phi(x)$ for some $T \in \mathbb{R}$. Using $T = -\infty$ for $\psi(\phi'_1, \phi'')$ and $T = \infty$ for $\psi(\phi'_2, \phi'')$, we find

$$\rho'(\phi'_1, \phi'_2) = Z(N, \phi'_1, \phi'_2). \tag{2.5}$$

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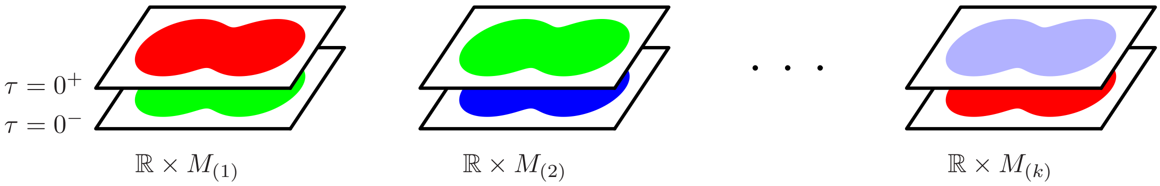

Figure 1: The replica method involves cutting the original manifold N along $\{\tau = 0\} \times M'$ and gluing k such cut copies of N along $\{\tau = 0\} \times M'$ to form the manifold N_k with k sheets. We identify $\{\tau_i = 0^-\} \times M'_{(i)}$ with $\{\tau_{i+1} = 0^+\} \times M'_{(i+1)}$ for $i = 1, \ldots, k-1$, and $\{\tau_k = 0^-\} \times M'_{(k)}$ with $\{\tau_1 = 0^+\} \times M'_{(1)}$. When $M = \mathbb{R}$ and Σ is a point P, the construction gives the 2-dimensional cone manifold $C_k = \mathbb{R}^+ \times S^1_k$, where S^1_k is the unit circle S^1 which is parametrized by $0 \leq \theta \leq 2\pi k$. The quantity $2\pi(1-k)$ is called the deficit angle. Note that C_k is the Riemann surface of the holomorphic function $z \mapsto z^k$.

The function $\varphi(\tau, x)$ has a discontinuity at $\tau = 0$ since $\varphi(0^-, x) = \phi'_1(x)$ and $\varphi(0^+, x) = \phi'_2(x)$. However, $\varphi(\tau, x)$ is continuous on the manifold $\tilde{N}_1$, which is defined as the manifold N with the cut along $\{\tau = 0\} \times M'$.

The kth power of the density matrix is

$$\rho'^k(\phi'_1, \phi'_{k+1}) = \int d\phi'_2 d\phi'_3 \cdots d\phi'_k \, \rho'(\phi'_1, \phi'_2)\rho'(\phi'_2, \phi'_3) \cdots \rho'(\phi'_k, \phi'_{k+1}). \tag{2.6}$$

Let $\mathbb{R} \times M_{(1)}, \ldots, \mathbb{R} \times M_{(k)}$ be k copies of $\mathbb{R} \times M$. By cutting every $\mathbb{R} \times M_{(i)}$ along $\{\tau_i = 0\} \times M'_{(i)}$ and gluing them in such a way that $\{\tau_i = 0^-\} \times M'_{(i)}$ is identified with $\{\tau_{i+1} = 0^+\} \times M'_{(i+1)}$ for $i = 1, \ldots, k-1$, we obtain the manifold $\tilde{N}_k$. See figure 1. This gives

$$\rho'^k(\phi'_1, \phi'_{k+1}) = Z(\tilde{N}_k, \phi'_1, \phi'_{k+1}). \tag{2.7}$$

Identifying $\{\tau_k = 0^-\} \times M'_{(k)}$ with $\{\tau_1 = 0^+\} \times M'_{(1)}$, we obtain the manifold N_k. This gives

$$\operatorname{tr} \rho'^k = Z(N_k) = \int_{C(N_k)} d\varphi \, \exp\left(-I(\varphi)\right), \tag{2.8}$$

which is a path integral over all closed curves in N_k. This quantity gives the entanglement entropy for ρ' via eq. (2.3). If instead we were to trace the density matrix over the degrees of freedom in M', we would obtained the reduced density matrix ρ'' for ϕ''. It is easy to show that the entanglement entropy for ρ'' is the same, $S' = S''$, and we denote the common value by S_Σ to emphasize its dependence on the surface Σ.

2.2 Heat kernel

To proceed with an explicit computation, we choose the free scalar field with the action

$$I(\varphi) = 2^{-1} \int_N \omega_N \varphi D_N \varphi, \tag{2.9}$$

where $D_N = \Delta_N + m^2$, Δ_N is the scalar Laplace operator for N, ω_N is the volume form for N, and m is the mass of the field φ. Performing a Gaussian integral, we find

$$Z(N_k) = Z_0^k (\det D_{N_k})^{-1/2}, \tag{2.10}$$

where Z_0 is a constant independent of k. Since D_{N_k} is a non-negative elliptic operator, we can define its determinant by

$$\log\det D_{N_k} - \log\det E_{N_k} = -\int_0^\infty dt\, t^{-1}\big(\operatorname{tr}\exp(-tD_{N_k}) - \operatorname{tr}\exp(-tE_{N_k})\big), \tag{2.11}$$

where E_{N_k} is any other non-negative elliptic operator on N_k. (To prove this equation, one writes the analogous equation relating eigenvalues of D_{N_k} and E_{N_k}.) The quantity $\exp(-tD_{N_k})$ is called the heat kernel of the operator D_{N_k}, and

$$K(t, D_{N_k}) = \operatorname{tr}\exp(-tD_{N_k}) \tag{2.12}$$

is its L^2 trace. We find $K(t, D_{N_k}) = \exp(-tm^2)K(t, \Delta_{N_k})$.

The integral over t in eq. (2.11) diverges for small t. To obtain a finite result, we replace the lower limit of integration over t by a regularization parameter λ^2 (an ultraviolet cutoff),

$$\int_{\lambda^2}^\infty dt\, t^{-1} K(t, D_{N_k}) = \operatorname{tr}\Gamma(0, \lambda^2 D_{N_k}). \tag{2.13}$$

Here Γ is the incomplete Gamma function which is given either by the integral representation

$$\Gamma(\alpha, z) = \int_z^\infty du\, u^{\alpha-1}\exp(-u) \tag{2.14}$$

or by the series representation

$$\Gamma(\alpha, z) = \Gamma(\alpha) - z^\alpha \sum_{j=0}^\infty \frac{(-z)^j}{(\alpha+j)j!}, \qquad \alpha \neq 0, -1, -2, \dots, \tag{2.15}$$

$$\Gamma(-l, z) = \frac{(-1)^l}{l!}(\psi(l+1) - \log z) - z^{-l}\sum_{\substack{j=0\\ j\neq l}}^\infty \frac{(-z)^j}{(-l+j)j!}, \qquad l = 0, 1, 2, \dots, \tag{2.16}$$

where $\psi(l+1) = -\gamma + \sum_{j=1}^l j^{-1}$ and γ is the Euler constant. We will later need the incomplete Gamma function for nonzero values of α as well.

We choose E_{N_k} to be a unit operator times a constant with the dimension of inverse length squared; this leads to vanishing of its contribution to the entropy. Similarly, the contribution from the constant Z_0^k vanishes. The regularized entropy becomes

$$S_\Sigma(\lambda) = 2^{-1}\lim_{k\to 1}\left(1 - \frac{\partial}{\partial k}\right)\operatorname{tr}\Gamma(0, \lambda^2 D_{N_k}). \tag{2.17}$$

We can factor the dependence of $S_\Sigma(\lambda)$ on Σ proceeding as follows. Locally, $N_k = C_k \times \Sigma$, where $C_k = \mathbb{R}^+ \times S_k^1$ is the 2-dimensional cone manifold, and S_k^1 is the unit circle S^1 which is parametrized by $0 \le \theta \le 2\pi k$. The quantity $2\pi(1-k)$ is called the deficit angle. Note that C_k is the Riemann surface of the holomorphic function $z \mapsto z^k$. Giving N_k a product metric, we find

$$\Delta_{N_k} = \Delta_{C_k} \otimes 1_\Sigma + 1_{C_k} \otimes \Delta_\Sigma, \tag{2.18}$$

which gives

$$K(t,\Delta_{N_k}) = K(t,\Delta_{C_k})K(t,\Delta_\Sigma). \tag{2.19}$$

This factorization reveals the special role played by the entropy for a point P, when $M=\mathbb{R}$, $\Sigma = P$,

$$S_P(\lambda) = 2^{-1}\int_{\lambda^2}^{\infty} dt\, t^{-1}\exp\left(-tm^2\right)C(t), \tag{2.20}$$

where

$$C(t) = \lim_{k\to 1}\left(1-\frac{\partial}{\partial k}\right)K(t,\Delta_{C_k}). \tag{2.21}$$

A simple computation gives the expression

$$S_\Sigma(\lambda) = -\int_{\lambda}^{\infty} d\mu\,\frac{\partial S_P(\mu)}{\partial\mu}K(\mu^2,\Delta_\Sigma), \tag{2.22}$$

in which the dependence on Σ is factored out. This equation leads to several properties of the entropy, which we now derive.

2.3 Several properties of entropy

1. Let (r,θ) be local polar coordinates for C_k, and $\xi>0$. Under the scaling transformation $(t,r,\theta)\mapsto(\xi^2 t,\xi r,\theta)$, we have $C_k\mapsto C_k$, $t\Delta_{C_k}\mapsto t\Delta_{C_k}$, and so $K(t,\Delta_{C_k})\mapsto K(t,\Delta_{C_k})$. This implies $C(t)=C=\text{const}$ and thus

$$S_P(\lambda) = 2^{-1}C\Gamma(0,\lambda^2 m^2). \tag{2.23}$$

   We will compute C in the next subsection and the appendix.

2. Let $(\tau = x^1, x^2)$ be local coordinates for C_k, let $(x^3,\dots,x^n)$ be local coordinates for Σ, and let $\xi>0$. Under the transformation $\lambda\mapsto\xi\lambda$, $x^j\mapsto\xi x^j$, $j=3,\dots,n$, we have $\Delta_\Sigma\mapsto\Delta_{\Sigma,\xi}=\xi^{-2}\Delta_\Sigma$ and $S_\Sigma(\lambda)\mapsto S_{\Sigma,\xi}(\xi\lambda)$, where

$$S_{\Sigma,\xi}(\xi\lambda) = -\int_{\xi\lambda}^{\infty} d\mu\,\frac{\partial S_P(\mu)}{\partial\mu}\left(\frac{\partial S_P(\xi^{-1}\mu)}{\partial(\xi^{-1}\mu)}\right)^{-1}\frac{\partial S_\Sigma(\xi^{-1}\mu)}{\partial(\xi^{-1}\mu)}. \tag{2.24}$$

   Using eq. (2.23), we find

$$S_{\Sigma,\xi}(\xi\lambda) = -\int_{\lambda}^{\infty} d\nu\,\exp\left(-(\xi^2-1)\nu^2 m^2\right)\frac{\partial S_\Sigma(\nu)}{\partial\nu}. \tag{2.25}$$

   It follows that $\lim_{\xi\to 0}(\partial S_{\Sigma,\xi}(\xi\lambda)/\partial\xi)=0$. Since $\mathrm{vol}\,(\Sigma)\mapsto\xi^{n-2}\,\mathrm{vol}\,(\Sigma)$, this implies

$$S_\Sigma(\lambda)\sim C'\lambda^{2-n}\,\mathrm{vol}\,(\Sigma),\quad \lambda\to 0, \tag{2.26}$$

   where $C'=\text{const}$.

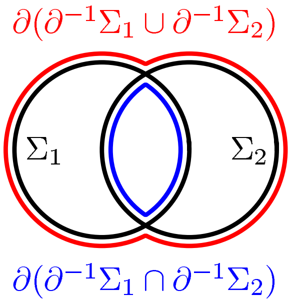

Figure 2: The hypersurfaces used in the formulation of the additivity property.

3. Since 0 is the smallest eigenvalue of Δ_Σ, we have $K(t, \Delta_\Sigma) \sim 1$, $t \to \infty$. This gives

$$S_\Sigma(\lambda) \sim 2^{-1} C \Gamma(0, \lambda^2 m^2), \quad \lambda \to \infty. \tag{2.27}$$

Interestingly, this coincides with the expression in eq. (2.23) for $S_P(\lambda)$ for arbitrary λ. This can be understood as the result of the physical scales of Σ becoming irrelevant as the cutoff λ tends to infinity.

4. Let Σ_1 and Σ_2 be closed hypersurfaces in M. Let ∂ and ∂^{-1} be operators defined by $\partial M'_i = \Sigma_i$ and $\partial^{-1}\Sigma_i = M'_i$, where M'_i is a part of M inside Σ_i for $i = 1, 2$. Being an integral over Σ, the quantity $K(t, \Delta_\Sigma)$ is linear in Σ. It follows that

$$K(t, \Delta_{\Sigma_1}) + K(t, \Delta_{\Sigma_2}) = K(t, \Delta_{\partial(\partial^{-1}\Sigma_1 \cup \partial^{-1}\Sigma_2)}) + K(t, \Delta_{\partial(\partial^{-1}\Sigma_1 \cap \partial^{-1}\Sigma_2)}), \tag{2.28}$$

and thus the entropy satisfies the additivity property

$$S_{\Sigma_1}(\lambda) + S_{\Sigma_2}(\lambda) = S_{\partial(\partial^{-1}\Sigma_1 \cup \partial^{-1}\Sigma_2)}(\lambda) + S_{\partial(\partial^{-1}\Sigma_1 \cap \partial^{-1}\Sigma_2)}(\lambda). \tag{2.29}$$

See figure 2. For an arbitrary system, the entanglement entropy satisfies the strong subadditivity property, which requires '$\geq$' instead of '$=$' in eq. (2.29).

2.4 Asymptotics

We now derive the asymptotic expansion of $S_\Sigma(\lambda)$ for $\lambda \to 0$. It is clear that this requires knowledge of the asymptotic behavior of $K(t, \Delta_\Sigma)$ for $t \to 0$. For an n-dimensional manifold L, such an asymptotic is given by

$$K(t, \Delta_L) \sim \sum_{l=0}^{\infty} t^{(l-n)/2} a_l(\Delta_L), \quad t \to 0, \tag{2.30}$$

where

$$a_l(\Delta_L) = \int_L \omega_L a_l(x_L, \Delta_L), \tag{2.31}$$

and $a_l(x_L, \Delta_L)$ are the heat kernel coefficients for Δ_L. The above factorization of $K(t, \Delta_{N_k})$ leads to

$$a_l(x_{N_k}, \Delta_{N_k}) = \sum_{j=0}^{l} a_j(x_{C_k}, \Delta_{C_k}) a_{l-j}(x_\Sigma, \Delta_\Sigma). \tag{2.32}$$

The coefficients $a_l(x_L, \Delta_L)$ are completely determined by the geometry of L. For a manifold without boundary, $a_l(x_L, \Delta_L) = 0$ for odd l. All coefficients $a_l(x_L, \Delta_L)$ are polynomials in the covariant derivatives of the Riemann tensor $(R_L)_{abcd}$, the Ricci tensor $(R_L)_{ab}$, and the scalar curvature R_L of L. Explicit expressions for several first coefficients are available in the literature (see, for example, ref. [20]). For example,

$$a_0(x_L, \Delta_L) =(4\pi)^{-n/2}, \tag{2.33}$$

$$a_2(x_L, \Delta_L) =(4\pi)^{-n/2}6^{-1}R_L, \tag{2.34}$$

$$\begin{aligned} a_4(x_L, \Delta_L) =&(4\pi)^{-n/2}360^{-1}\Big(-12\Delta_L R_L + 5R_L^2 - 2\sum_{a,b}(R_L)_{ab}(R_L)_{ab} \\ &+ 2\sum_{a,b,c,d}(R_L)_{abcd}(R_L)_{abcd}\Big). \end{aligned} \tag{2.35}$$

It has been shown [22] that the only nonzero heat kernel coefficients for C_k are

$$a_0(x_{C_k}, \Delta_{C_k}) = (4\pi)^{-1}, \tag{2.36}$$

$$a_2(x_{C_k}, \Delta_{C_k}) = (4\pi)^{-1}6^{-1}4\pi(1-k)\delta_{C_k}, \tag{2.37}$$

where δ_{C_k} is the delta function at the origin of C_k. This gives $C = 6^{-1}$. (In the appendix, we derive this result.) The regularized entropy becomes

$$S_P(\lambda) = 12^{-1}\Gamma(0, \lambda^2 m^2), \tag{2.38}$$

$$S_\Sigma(\lambda) = 12^{-1}\,\mathrm{tr}\,\Gamma(0, \lambda^2 D_\Sigma). \tag{2.39}$$

In terms of the integrated heat kernel coefficients of Σ, we find

$$S_\Sigma(\lambda) \sim 12^{-1}\sum_{l=0}^{\infty} m^{n-l-2}\Gamma\big((2+l-n)/2, (\lambda m)^2\big)a_l(\Delta_\Sigma), \quad \lambda \to 0. \tag{2.40}$$

This asymptotic expansion is our main result. For $\lambda \to 0$, $S_\Sigma(\lambda)$ depends only on the spectral properties of the operator $\lambda^2 D_\Sigma$. Equivalently, the entropy depends only on parameters m, λ, and on geometric invariants associated with Σ. The asymptotic expansion for $S_\Sigma(\lambda)$ involves $\log \lambda m$ and the powers of λm. The leading term in the entropy is

$$S_P(\lambda) \sim 12^{-1}\big(-2\log\lambda m - \gamma\big), \quad \lambda \to 0, \tag{2.41}$$

$$S_\Sigma(\lambda) \sim 12^{-1}(n/2-1)^{-1}\lambda^{2-n}(4\pi)^{1-n/2}\,\mathrm{vol}\,(\Sigma), \quad \lambda \to 0. \tag{2.42}$$

The term of order $\lambda^{-n}\,\mathrm{vol}\,(N)$ in $S_\Sigma(\lambda)$ is absent; it would be the extensive contribution to the entropy.

We remark on the case $n = 2$. (See also ref. [15] for a similar discussion.) The entanglement entropy for a critical 2-dimensional CFT with the central charge c is asymptotically $S \sim (c/3)\log(\ell/\lambda)$, where ℓ is the size of the system [11]. For a massive theory with the correlation length ξ, the entropy becomes $S \sim (c/6)\nu\log(\xi/\lambda)$ for $\ell \gg \xi$, where ν is the number of components of (zero dimensional) Σ. Setting $c = 1$, $\nu = 1$, $\xi \sim m^{-1}$, we recover eq. (2.41).

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3. Conclusion

We have calculated the asymptotic expansion of the entanglement entropy for a free scalar field in arbitrary background geometry. The expansion parameter is the ultraviolet cutoff λ which is needed to regularize the entropy. We have found that the entropy depends only on geometric invariants associated with the boundary surface Σ. The extensive contribution to the entropy, the term of order $\lambda^{-n}\,\mathrm{vol}\,(N)$, is absent. The leading term is proportional to $\lambda^{2-n}\,\mathrm{vol}\,(\Sigma)$.

We have considered a situation with spatial curvature only and with time included in only a trivial way as a product. Our calculation does not utilize a spacetime that is a solution of Einstein's equations. Further research may involve extending this calculation to cases involving spacetime curvature. Another interesting direction to pursue is to include interactions since this, at least intuitively, can potentially change the area dependence of the entropy. The studies of interactions have been mostly limited to CFTs, but their role in QFT remains largely unexplored. These cases involve gravity more explicitly and may make a connection between holographic entropy and entanglement entropy more obvious.

Acknowledgments

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A. Heat kernel coefficients for the cone

Here we compute the heat kernel coefficients for the cone C_k. Let $\xi > 0$. Under the transformation $(t, r, \theta) \mapsto (\xi^2 t, \xi r, \theta)$, we have $K(t, \Delta_{C_k}) \mapsto K(t, \Delta_{C_k})$, $a_l(\Delta_{C_k}) \mapsto \xi^{2-l} a_l(\Delta_{C_k})$. Since C_k does not have a length scale associated with it, this implies $a_0(\Delta_{C_k}) = \infty$, $a_2(\Delta_{C_k}) = \text{const}$, $a_l(\Delta_{C_k}) = 0$, $l \geq 4$. $a_0(x_{C_k}, \Delta_{C_k})$ is given by eq. (2.33), and to compute $a_2(x_{C_k}, \Delta_{C_k})$ from eq. (2.34), we need to know the scalar curvature of C_k, with computation of which we now proceed.

C_k is singular at $r = 0$ if $k \neq 1$. We consider it as a limit $C_k = \lim_{\varepsilon\to 0} C_{k,\varepsilon}$, where $C_{k,\varepsilon}$ is a regular manifold. On $C_{k,\varepsilon}$ we take an orthonormal frame $(\omega^1, \omega^2) = (f dr, r d\theta)$, where the regularization function $f(k, r, \varepsilon)$ is an arbitrary smooth function satisfying conditions $\lim_{r\to 0} f = k$, $\lim_{\varepsilon\to 0} f = 1$. An example of such a function is $f = \left(k^2 + (1 - k^2)(qr)^\varepsilon\right)^{1/2}$, where $\varepsilon \geq 0$ and $q > 0$ is an arbitrary constant with the dimension of inverse length. In what follows, we proceed with arbitrary f satisfying the above conditions.

Let ω and Ω be the 2×2 antisymmetric matrices of connection and curvature forms. Cartan's equations for $C_{k,\varepsilon}$,

$$\omega^1{}_2 \wedge r d\theta = 0, \tag{A.1}$$

$$dr \wedge d\theta + \omega^2{}_1 \wedge f dr = 0, \tag{A.2}$$

$$d\omega^1{}_2 = \Omega^1{}_2, \tag{A.3}$$

have the solution

$$\omega^1{}_2 = -f^{-1}d\theta, \tag{A.4}$$

$$\Omega^1{}_2 = rfg dr \wedge d\theta, \tag{A.5}$$

where $g = r^{-1}f^{-3}(\partial f/\partial r)$. The nonzero components of the Ricci tensor are $(R_{C_{k,\varepsilon}})_{11} = (R_{C_{k,\varepsilon}})_{22} = g$, and the scalar curvature is $R_{C_{k,\varepsilon}} = 2g$.

To obtain non-regularized quantities, we consider the limit $\varepsilon \to 0$. Using $\lim_{r\to 0} f = k$, $\lim_{\varepsilon\to 0} f = 1$, $\omega_{C_{k,\varepsilon}} = rf dr \wedge d\theta$, for an arbitrary function $h(r)$ satisfying $h(\infty) = 0$, we find

$$\lim_{\varepsilon\to 0}\int_{C_{k,\varepsilon}} \omega_{C_{k,\varepsilon}} gh = 2\pi k \lim_{\varepsilon\to 0}\left(-f^{-1}h\big|_{r=0}^{r=\infty} + \int_0^\infty dr\, f^{-1}(\partial h/\partial r)\right) = 2\pi(1-k)h(0). \tag{A.6}$$

This implies $\lim_{\varepsilon\to 0} g = 2\pi(1-k)\delta_{C_k}$, where δ_{C_k} is the delta function at the origin of C_k. Thus, $R_{C_k} = 4\pi(1-k)\delta_{C_k}$, the only nonzero heat kernel coefficients for C_k are

$$a_0(x_{C_k}, \Delta_{C_k}) = (4\pi)^{-1}, \tag{A.7}$$

$$a_2(x_{C_k}, \Delta_{C_k}) = (4\pi)^{-1}6^{-1}4\pi(1-k)\delta_{C_k}, \tag{A.8}$$

so that we may identify $C = 6^{-1}$. Alternatively, since $C(t)$ in eq. (2.21) is a constant, we can compute it by taking the limit $t \to 0$ in the expansion

$$C(t) \sim \lim_{k\to 1}\left(1 - \frac{\partial}{\partial k}\right)\sum_{l=0}^{\infty} t^{(l-2)/2} a_l(\Delta_{C_k}), \quad t \to 0. \tag{A.9}$$

Since the coefficients $a_0(x_{C_k}, \Delta_{C_k})$, $a_l(x_{C_k}, \Delta_{C_k})$, $l \geq 4$ do not contribute to C, we find $C = 6^{-1}$.

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